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Quantisation of limit cycles in a P representation of a dissipative driven anharmonic oscillator

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Abstract. The non-linear polarisability model of dispersive optical bistability with periodic driving field is shown to have non-trivial limit cycle solutions in phase space. The model is quantised rigorously in the positive P representation. For a particular (but typical) limit cycle, the associated steady state probability distribution is shown to be compatible with a bivariate Gaussian. The inverse widths of the Gaussian are calculated as functions of position in both phase space and time.

1. Introduction

Recently there has been interest in dissipative and Hamiltonian systems with a small number of degrees of motion which have complicated periodic and aperiodic classical orbits [1]. The quantum analogue of these motions is now attracting attention although mostly in Hamiltonian systems. In fact most non-linear systems are known to show complicated orbits. Hence there is an enormous choice of systems whose quantum analogues can be studied. We will study a model which arises naturally from the theory of dispersive optical bistability [2]. Its advantage is that its quantisation can be treated in a satisfactory manner.

The classical system is described by

$$\dot{p} = -\gamma p - \omega^2 q - \lambda(p^2 + \omega^2 q^2)\omega^2 q + \mathcal{E} \cos \omega_d t \quad (1)$$

$$\dot{q} = p + \lambda(p^2 + \omega^2 q^2)p \quad (2)$$

where p is the position and q the momentum of an oscillator. In the absence of the periodic driving term $\mathcal{E} \cos \omega_d t$ and the damping γ these are just Hamilton's equations associated with the Hamiltonian

$$H = \frac{1}{2}(p^2 + \omega^2 q^2) + \frac{1}{4}\lambda(p^2 + \omega^2 q^2)^2 \quad (3)$$

where λ is a coupling constant and ω is a frequency. Without the damping and driving, the system is trivially integrable, and when quantised the standard number states [3] form a complete set of eigenstates. If there is too much damping the motion will be simple and lie in the vicinity of the origin. The non-linearity of the oscillator implies that the effective frequency of the oscillator depends on its amplitude. With driving and a reasonable amount of damping complicated classical orbits may arise. We have only found quite involved limit cycles, but no chaotic trajectories.

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The quantum analogue of (1) and (2) can be derived from the Hamiltonian H'

$$H' = \hbar\omega(a^\dagger a + \frac{1}{2}) + \lambda \hbar^2 \omega^2 (a^\dagger a + \frac{1}{2})^2 + (\hbar/2\omega)^{1/2} \mathcal{E}(a + a^\dagger) \cos \omega_d t$$

$$+ \sum_i \hbar\omega_i (\Gamma_i^\dagger \Gamma_i + \frac{1}{2}) + \sum_i (g_i a \Gamma_i^\dagger + g_i^* a^\dagger \Gamma_i) \quad (4)$$

where a and a^\dagger are the annihilation and creation operators of the oscillator of interest, Γ_i and Γ_i^\dagger are similar quantities for the i th reservoir oscillator of natural frequency ω_i and g_i is the coupling strength of the i th component of the reservoir with the anharmonic oscillator. This is a standard way of consistently introducing dissipation into quantum systems. The resulting master equation [4] for the density matrix is

$$\partial\rho/\partial t = -i\omega[a^\dagger a, \rho] - i\lambda \hbar\omega^2 [(a^\dagger a)^2 + a^\dagger a, \rho] + \gamma(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a)$$

$$+ i(2\omega\hbar)^{-1/2} \mathcal{E}[\rho, a + a^\dagger] \cos \omega_d t. \quad (5)$$

This operator equation needs to be transformed into a c -number set. In a number state basis we have an infinite hierarchy of coupled linear ordinary differential equations. In general a criterion needs to be introduced to truncate the infinite hierarchy. We shall follow a different approach. It is useful to express (5) as a Fokker-Planck equation (FPE) for a positive P representation [5]. The moments of P give normally ordered expectation values and it can be interpreted as a classical distribution on an enlarged phase space. A particularly attractive feature of (5) is that the corresponding FPE is exact. For most systems the FPE is obtained from the master equation only after a system size expansion [6].

The FPE for P is given by

$$\partial P/\partial t = (\partial/\partial\alpha^{(1)})[i\alpha^{(1)} + (\gamma/\omega)\alpha^{(1)} + 2i\varepsilon(\alpha^{(1)} + (\alpha^{(1)})^2)\alpha^{(2)}]$$

$$+ i\mathcal{E}\omega^{-1}(2\omega\hbar)^{-1/2} \cos(\omega_d/\omega)\tau]$$

$$+ (\partial/\partial\alpha^{(2)})[-i\alpha^{(2)} + (\gamma/\omega)\alpha^{(2)} - 2i\varepsilon(\alpha^{(2)} + \alpha^{(1)}(\alpha^{(2)})^2)]$$

$$- i\mathcal{E}\omega^{-1}(\hbar/2\omega)^{1/2} \cos(\omega_d/\omega)\tau]$$

$$- i\varepsilon[(\partial^2/\partial\alpha^{(1)2})(\alpha^{(1)})^2 - (\partial^2/\partial\alpha^{(2)2})(\alpha^{(2)})^2] \quad (6)$$

where τ is ωt and ε is $\lambda\hbar\omega$.

In the positive P representation, although $\alpha^{(1)}$ corresponds to a and $\alpha^{(2)}$ to a^\dagger , no explicit constraint that $\alpha^{(1)}$ is conjugate to $\alpha^{(2)}$ is incorporated. This gives rise to the extended phase space. It is convenient to transform (6) into a complex Ito-Langevin stochastic differential equation (SDE)

$$d\alpha^{(i)} = A_i d\tau + d_{ij} dW_j \quad i = 1, 2. \quad (7)$$

Here

$$A = \begin{pmatrix} -(i\alpha^{(1)} + \gamma\omega^{-1}\alpha^{(1)} + 2i\varepsilon(\alpha^{(1)} + \alpha^{(2)}(\alpha^{(1)})^2) + i\hat{\mathcal{E}} \cos(\omega_d/\omega)\tau) \\ i\alpha^{(2)} - \gamma\omega^{-1}\alpha^{(2)} + 2i\varepsilon(\alpha^{(2)} + \alpha^{(1)}(\alpha^{(2)})^2) + i\hat{\mathcal{E}} \cos(\omega_d/\omega)\tau \end{pmatrix} \quad (8)$$

where

$$\hat{\mathcal{E}} = \mathcal{E}\omega^{-1}(2\omega\hbar)^{-1/2}$$

and

$$d = (-2i\varepsilon)^{1/2} \begin{pmatrix} \alpha^{(1)} \\ i\alpha^{(2)} \end{pmatrix}. \quad (9)$$

W_j is a Wiener process, those with different j being independent of each other. Equation (7) will first be solved numerically. Schematically this equation can be written as

$$d\alpha^{(i)}(\tau) = f_i(\alpha^{(1)}(\tau), \alpha^{(2)}(\tau)) d\tau + e_i\alpha^{(i)}(\tau) dW_i(\tau). \tag{10}$$

No summation convention for repeated indices is in force here. A lowest-order algorithm for solving this is

$$\alpha^{(i)}(\tau) = \alpha^{(i)}(0) + f_i(\alpha^{(1)}(0), \alpha^{(2)}(0))\tau + e_i\alpha^{(i)}(0)(W_i(\tau) - W_i(0)) \tag{11}$$

which is a better approximation the smaller τ is. This corresponds to the Euler method for conventional differential equations, and is found to be numerically unstable for time steps as short as 10^{-2} even when the damping is as large as $\gamma = 0.5$. These instabilities occur, as expected, more readily in cases of light damping. To overcome these difficulties we have used a second-order Taylor expansion for the stochastic process $\alpha^{(i)}$. This gives

$$\begin{aligned} \alpha^{(i)}(\tau) = & \alpha^{(i)}(0) + f_i(\alpha^{(1)}(0), \alpha^{(2)}(0))\tau + e_i\alpha^{(i)}(0) W_i(\tau) \\ & + \sum_{j=1}^2 f_{i,j}(\alpha^{(1)}(0), \alpha^{(2)}(0))e_j\alpha^{(j)}(0) \int_0^\tau W_j(s) ds \\ & + e_i f_i(\alpha^{(1)}(0), \alpha^{(2)}(0)) \int_0^\tau s dW_i(s) + e_i^2\alpha^{(i)}(0) \int_0^\tau W_i(s) dW_i(s) \\ & + e_i^3\alpha^{(i)}(0) \int_0^\tau dW_i(s) \int_0^s dW_i(s') W_i(s') \\ & + \frac{1}{2} \sum_{j=1}^3 f_{i,j}(\alpha^{(1)}(0), \alpha^{(2)}(0))f_j(\alpha^{(1)}(0), \alpha^{(2)}(0))\tau^2 + O(\tau^{5/2}). \end{aligned} \tag{12}$$

Essentially all previous work on SDE has been based on (11), and most authors have considered single variable SDE. Helfand [7] and Greenside and Helfand [7] considered coupled SDE with only additive noise. Rao *et al* [7], however, treated a one-variable SDE with multiplicative noise and produced an algorithm equivalent to a fourth-order Taylor expansion for the stochastic process. Coupled SDE with multiplicative noise, however, have not been tackled before beyond the Euler algorithm. The derivation of (12) is given in appendix 1.

A complementary approach will also be sketched which is valid in the semiclassical limit [8] and it will be found that the essential features obtained using the more general stochastic differential equations are compatible with those in the semiclassical limit.

The anharmonic system has many distinct limit cycles. We show three in the figures; we will concentrate on the limit cycle in figure 3 because it is non-trivial, yet without being too complicated. The system parameters are taken to be $\gamma = 0.5$, $\varepsilon = 10^{-3}$, $\omega_a\omega^{-1} = 1$, $\hat{\xi} = 200$. We will deal exclusively with the variables $\text{Re}(\frac{1}{2}(\alpha^{(1)} + \alpha^{(2)}))$, \hat{q} say, and $\text{Im}(\frac{1}{2}(\alpha^{(1)} - \alpha^{(2)}))$, \hat{p} say. It is easy to show that in terms of these variables the uncertainty relation has the form

$$\Delta(\hat{q})\Delta(\hat{p}) \geq \frac{1}{4} \tag{13}$$

where $\Delta(x)$ denotes the standard deviation of the variable x . \hat{q} is proportional to q and \hat{p} to p (at least in the sense that their moments are related). The limit cycles are just plots of $\langle \hat{q} \rangle$ (the expectation value of \hat{q}) against that of $\langle \hat{p} \rangle$. The statistics around the mean are given by the following normally ordered expectation values M_{rs} :

$$M_{rs} = \langle :(\hat{q} - \langle \hat{q} \rangle)^r (\hat{p} - \langle \hat{p} \rangle)^s : \rangle \tag{14}$$

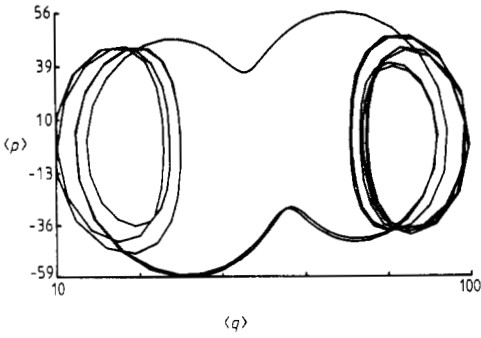


Figure 1. Limit cycle with $\gamma = 0.5$, $\varepsilon = 10^{-3}$, $\omega_d \omega^{-1} = 2.0$, $\hat{\mathcal{E}} = 1000$.

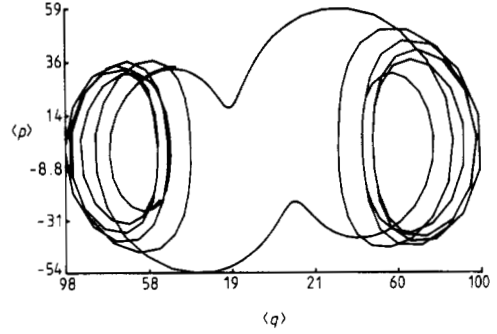


Figure 2. Limit cycle with $\gamma = 0.5$, $\varepsilon = 10^{-3}$, $\omega_d \omega^{-1} = 1.3$, $\hat{\mathcal{E}} = 1000$.

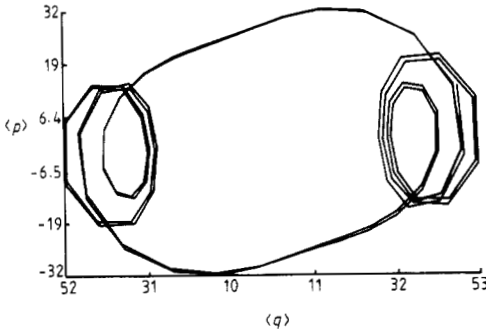


Figure 3. Limit cycle with $\gamma = 0.5$, $\varepsilon = 10^{-3}$, $\omega_d \omega^{-1} = \pi/3$, $\hat{\mathcal{E}} = 200$.

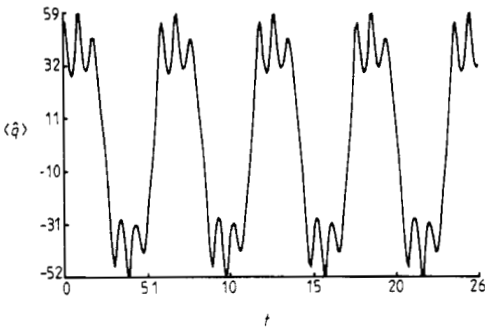


Figure 4. Time series for $\langle \hat{q} \rangle$.

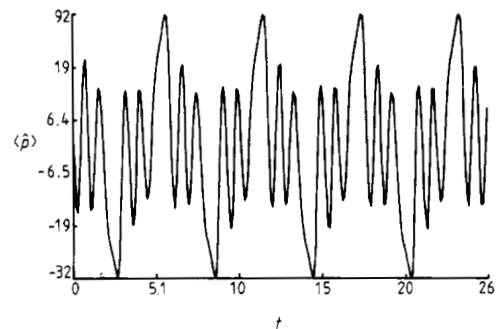


Figure 5. Time series for $\langle \hat{p} \rangle$.

the colons denoting normal ordering. We estimate the values of the moments from an ensemble of 500 trajectories. All the trajectories are started from the same initial condition, and sums (over ensembles) like $\Sigma \alpha^{(1)}$, $\Sigma \alpha^{(1)2}$, $\Sigma \alpha^{(1)} \alpha^{(2)}$, etc, are recorded by the program. This allows us to build up our statistics over many runs of the program. The moments are estimated from these sums. No time averaging is employed, and hence no assumptions about ergodicity are needed. The calculation of the moments for too high an order is unreliable owing to insufficient statistics. We have calculations for $(r + s) \leq 4$. An ensemble of 500 was mostly used. For a Gaussian process it is well

known that

$$\begin{aligned}
 M_{12} &= M_{21} = 0 \\
 M_{40} &= 3(M_{20})^2 \\
 M_{04} &= 3(M_{02})^2 \\
 M_{31} &= 3M_{20}M_{11} \\
 M_{13} &= 3M_{02}M_{11}.
 \end{aligned}
 \tag{15}$$

It turns out that within statistical uncertainties the positive P distribution has moments up to fourth order which are consistent with Gaussian statistics, and this is a property which holds independent of time. Since there are no points for which the derivatives of the drift vanish on the limit cycles that we are considering, it is to be expected that sufficiently near the limit cycles the probability distribution should be Gaussian. In fact this is a useful check on the consistency of the statistical procedures used. However the determination of the shape of the Gaussian along a circuit of the limit cycle is non-trivial. The time series for M_{20} and M_{02} in figures 6 and 7 are compatible with being periodic with a frequency twice that for the evolution of $\langle \hat{q} \rangle$ and $\langle \hat{p} \rangle$. This was found rigorously in the semiclassical approach, and is related to the inversion symmetry of the limit cycles considered around their centres.

The uncertainty relation can be expressed in the form

$$(M_{02}M_{20})^{1/2} \geq \frac{1}{4}
 \tag{16}$$

and we have plotted $(M_{20}M_{02})^{1/2}$ in figure 8, which is of course also periodic. A minimum uncertainty state is not achieved along the limit cycle and so we are not near a semiclassical regime. We see that the variances of \hat{q} and \hat{p} oscillate as the system traverses the limit cycle. This does not necessarily mean that the size of the distribution is varying in time; it could be due to a distribution of constant size rotating in phase space. If the distribution was elongated the projections onto the axes would appear to oscillate. We can exclude this possibility by diagonalising the covariance matrix and examining the major and minor axes of the distribution as a function of time. We find that the diagonalised quantities also oscillate. Furthermore the distribution is not, in general, very elongated since the typical ratio of major to minor axis length is only about 1.2:1. Finally, the minor axis length is everywhere greater than $\epsilon^{1/2}$, so the distributions do not show any sign of squeezing.

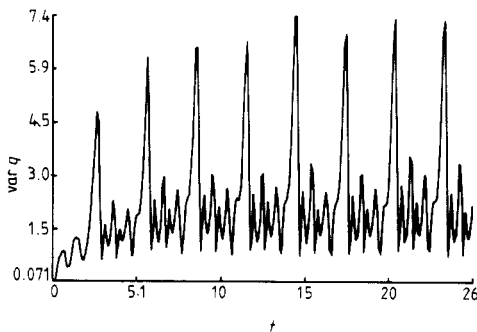


Figure 6. Time series for M_{20} .

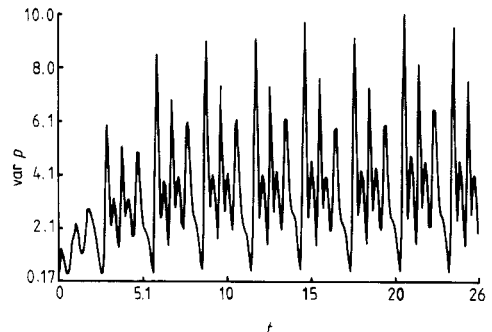


Figure 7. Time series for M_{02} .

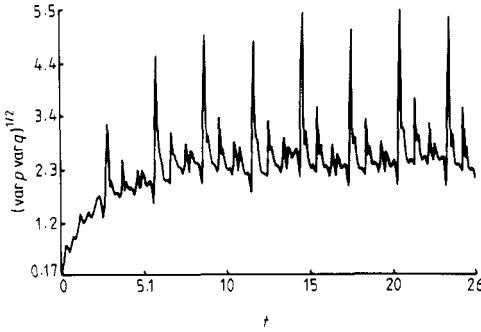


Figure 8. Time series for $(M_{20}M_{02})^{1/2}$.

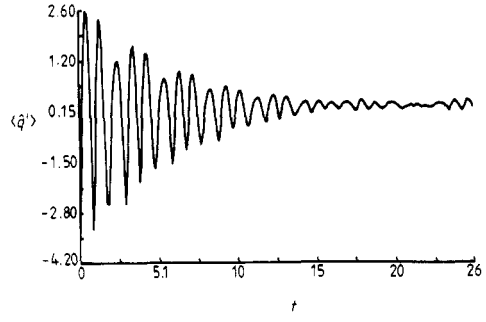


Figure 9. Time series for $\langle \hat{q}' \rangle$.

The positive P representation has auxiliary unphysical variables and it is interesting to check the variances in these. Paralleling the development for the physical degrees of freedom we introduce the notation \hat{q}' and \hat{p}' where

$$\begin{aligned} \hat{q}' &= \frac{1}{2} \text{Re}(\alpha^{(1)} - \alpha^{(2)*}) \\ \hat{p}' &= \frac{1}{2} \text{Im}(\alpha^{(1)} - \alpha^{(2)*}) \end{aligned} \tag{17}$$

and the moments M'_{rs} are given by

$$M'_{rs} = \langle :(\hat{q}' - \langle \hat{q}' \rangle)^r (\hat{p}' - \langle \hat{p}' \rangle)^s : \rangle. \tag{18}$$

As expected $\langle \hat{q}' \rangle$ and $\langle \hat{p}' \rangle$ are compatible with zero. The calculations of the variances M'_{20} and M'_{02} show a remarkable similarity in structure to M_{02} and M_{20} , respectively, in particular for the periodicity. Owing to the dissipative nature of the system, there is no conservation of the extended phase space volume.

Although the SDE approach is powerful it is limited by statistics. Consequently it would be satisfying if there was a complementary analysis showing qualitatively similar behaviour. In the semiclassical regime we can calculate the moments discussed above, but using ordinary differential equations. Indeed there is much corroboration with the results already found. Since the semiclassical analysis is based on a study of ordinary differential equations the accuracy of the calculation of moments is not limited by statistics. If the FPE of (6) were rewritten in terms of the real and imaginary parts of $\bar{\alpha}^{(1)}$ and $\bar{\alpha}^{(2)}$ as described in appendix 2 it would have the canonical form [7]

$$\partial P / \partial t = -(\partial / \partial x_i)(A_i P) + \epsilon(\partial^2 / \partial x_i \partial x_j)(D_{ij} P) \tag{19}$$

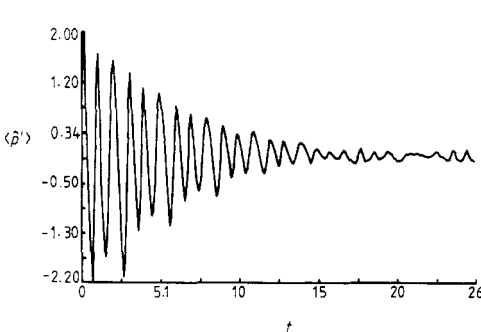


Figure 10. Time series for $\langle \hat{p}' \rangle$.

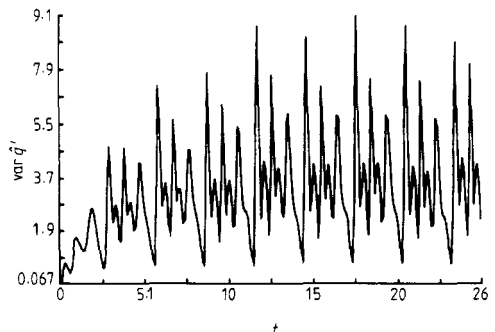


Figure 11. Times series for M'_{20} .

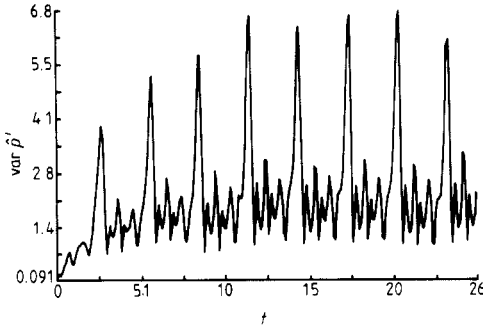


Figure 12. Time series for M_{02} .

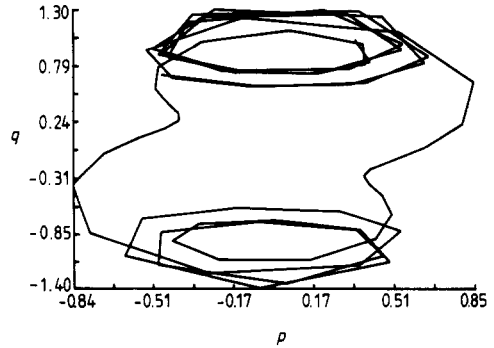


Figure 13. The motion in phase space of the peak of the probability distribution in the semiclassical limit for $\gamma = 0.5$, $\lambda = 10$, $\omega_d = 2.0$ and $\epsilon = 14.1$.

(with (x_i) , $1 \leq i \leq n$ denoting real variables). Assuming that there is a solution of the form

$$P = P_0 \exp(-W/\epsilon) \tag{20}$$

it can be shown that W_0 satisfies

$$\partial W_0/\partial t = -A_i(\partial W_0/\partial x_i) - D_{ij}(\partial \dot{W}_0/\partial x_i)(\partial W_0/\partial x_j) \tag{21}$$

where

$$W = W_0 + O(\epsilon).$$

The semiclassical limit is equivalent to $\epsilon \rightarrow 0$.

Equation (21) is obtained by equating terms of $O(1)$ when (20) has been used in (19). These are first-order differential equations and so can be solved by the standard method of characteristics [8, 10]. The equations for the characteristics have the form

$$dx_i/dt = A_i + 2D_{ij}P_j \tag{22}$$

$$dW/dt = (A_j + 2D_{ji}P_i)P_j + C \tag{23}$$

$$dP_i/dt = -[P_j(\partial/\partial x_i)A_j + ((\partial/\partial x_i)D_{ij})P_iP_j] \tag{24}$$

$$C = -A_iP_i - D_{ij}P_iP_j. \tag{25}$$

The method of solution of (22)-(25) has been given in [8] and [9]. However for clarity we will sketch here the main steps. Given an initial peaked distribution $W(x_1, x_2, x_3, x_4)$ (where the $\{x_i\}$ are $\{\bar{\alpha}_x^{(1)}, \bar{\alpha}_x^{(2)}, \bar{\alpha}_y^{(1)}, \bar{\alpha}_y^{(2)}\}$) it is possible to take a uniform grid of points around the peak of W in the four-dimensional X space. For each such point, (22)-(25) can be solved with the initial condition given by W and $\partial W/\partial x_i$. The time evolution gives a characteristic. At a time t_0 the values of W , $\partial W/\partial x_i$ and x_i determine the distribution P at $\{x_i\}$. Hence, if the characteristics are found for the grid of initial points, then, at time t , W is determined at the grid of endpoints of the characteristics. However a naive application of this procedure is not useful since the characteristics have a tendency to run off to infinity where the probability is negligible. In order to obtain the distribution near its peak it is necessary to propagate the characteristics for a short time (so that the grid of values found for W are still representative of the peak of the distribution). The resulting W is fitted using a suitable basis of functions. The

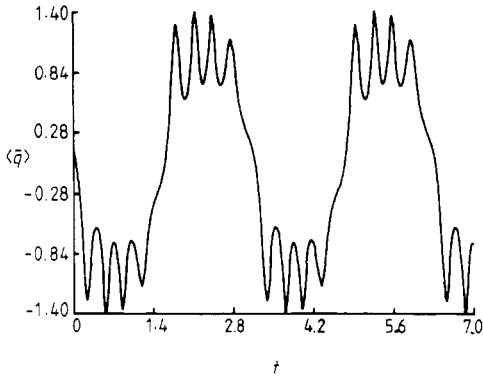


Figure 14. Time series of \bar{q} projection of orbit of centre of distribution.

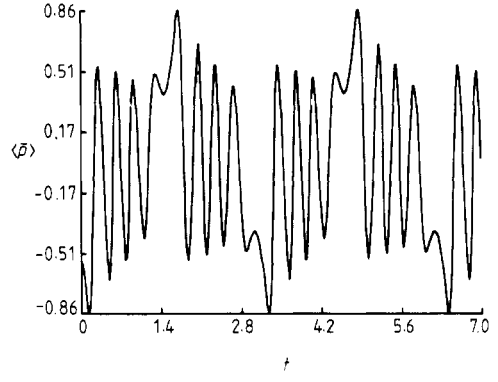


Figure 15. Time series of \bar{p} projection of orbit of centre of distribution.

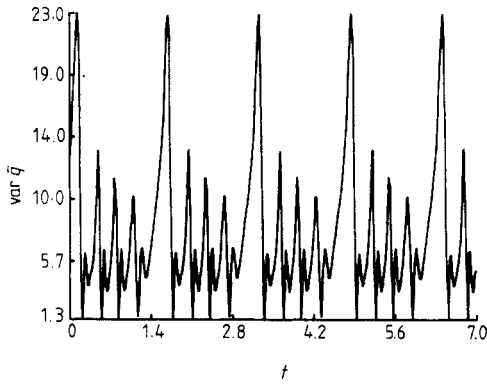


Figure 16. Time series for $\text{var } \bar{q}$.

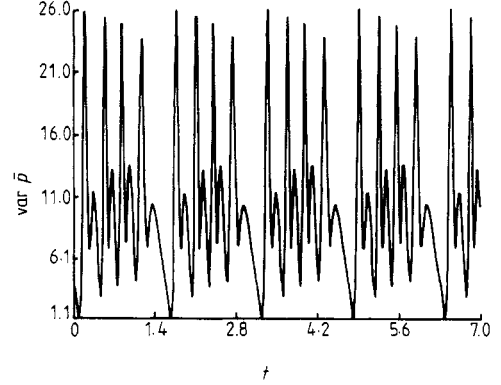


Figure 17. Time series for $\text{var } \bar{p}$.

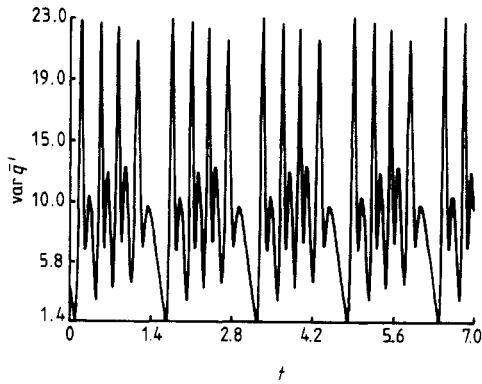


Figure 18. Time series for $\text{var } \bar{q}'$.

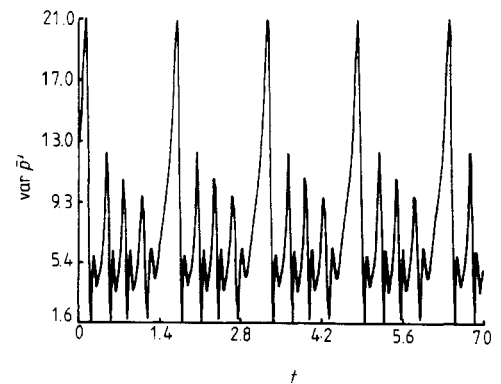


Figure 19. Time series for $\text{var } \bar{p}'$.

procedure is then repeated. In this way the probability distribution can be found as a function of time.

For our particular case W is taken to have the expansion

$$W = \sum_{i,j,k,l=1}^2 C_{ijkl} \bar{\alpha}_{ij} \bar{\alpha}_{kl} \tag{26}$$

where

$$\bar{\alpha}^{(j)} = \bar{\alpha}_{j1} + i \bar{\alpha}_{j2} \quad j = 1, 2. \tag{27}$$

Unless any of the coefficients C_{ijkl} are anomalously small the non-Gaussian contributions to W are negligible in the semiclassical limit. The Hamilton-Jacobi equations give C_{ijkl} as a function of t . It is convenient to arrange these coefficients into a matrix Γ

$$\Gamma = \begin{pmatrix} C_{1111} & \frac{1}{2}C_{1112} & \frac{1}{2}C_{1121} & \frac{1}{2}C_{1122} \\ \frac{1}{2}C_{1112} & C_{1212} & \frac{1}{2}C_{1221} & \frac{1}{2}C_{1222} \\ \frac{1}{2}C_{1121} & \frac{1}{2}C_{2112} & C_{2121} & \frac{1}{2}C_{2122} \\ \frac{1}{2}C_{1122} & \frac{1}{2}C_{1222} & \frac{1}{2}C_{2122} & C_{2222} \end{pmatrix}. \tag{28}$$

The coefficients C_{ijkl} of course have the symmetry property

$$C_{ijkl} = C_{klij}. \tag{29}$$

In order to compare results with those of the SDE approach it is necessary to consider Γ^{-1} , the inverse of Γ . Γ^{-1} is the matrix of covariances. If $\delta\bar{\alpha}_{ij}$ denotes the deviation of $\bar{\alpha}_{ij}$ from its determination value then

$$(\Gamma^{-1})_{ijkl} = \langle : \delta\bar{\alpha}_{ij} \delta\bar{\alpha}_{kl} : \rangle. \tag{30}$$

We can, as before, introduce the quantities

$$\bar{q} = \frac{1}{2}(\bar{\alpha}_{11} + \bar{\alpha}_{21}) \tag{31}$$

$$\bar{p} = \frac{1}{2}(\bar{\alpha}_{12} - \bar{\alpha}_{22}) \tag{32}$$

$$\bar{q}' = \frac{1}{2}(\bar{\alpha}_{11} - \bar{\alpha}_{21}) \tag{33}$$

$$\bar{p}' = \frac{1}{2}(\bar{\alpha}_{12} + \bar{\alpha}_{22}) \tag{34}$$

and calculate $\langle \bar{q} \rangle$, $\text{var } \bar{p}$, $\text{var } \bar{q}$, $\text{var } \bar{p}'$ and $\text{var } \bar{q}'$. The calculations show that the Gaussian ansatz of (26) is a good representation for the probability distribution. Moreover the periodic motion of $\langle \bar{q} \rangle$ and $\langle \bar{p} \rangle$ has half the frequency of the motion of $\text{var } \bar{q}$ and $\text{var } \bar{p}$. As noticed in the SDE calculation $\text{var } \{\bar{q}\}$ and $\text{var } \{\bar{p}'\}$ are similar. The calculation is summarised in figures 13-19. These results are consistent with the findings of the SDE approach and show that the semiclassical and fully quantal situations have qualitatively similar behaviour.

2. Conclusions

We have constructed the positive P probability distribution in the parameter region where the non-linear polarisability model has non-trivial limit cycles. The importance of this work lies in the following points:

- (i) understanding the nature of the positive P representation;
- (ii) as a step towards quantising non-trivial attractors (which exist in classical phase space) when there are no analytic solutions for these attractors;

(iii) going beyond the universally used algorithms for solving SDE for coupled non-linear SDE with (albeit a specific form of) multiplicative noise;

(iv) the quantisation is exact in principle unlike essentially all non-trivial dissipative systems that have been studied so far; and

(v) comparing a fully quantum mechanical treatment with semiclassical methods.

Specifically, the SDE and semiclassical approach have shown very good qualitative agreement concerning the behaviour of $\langle p \rangle$, $\langle q \rangle$, $\langle p' \rangle$, $\langle q' \rangle$, $\text{var } p$, $\text{var } q$, $\text{var } p'$ and $\text{var } q'$. The variances in the unphysical degrees of freedom show not only similar magnitudes to those in the physical degrees of freedom but $\text{var } p \approx \text{var } q'$ and $\text{var } q \approx \text{var } p'$. The moments are compatible with the positive P distribution being Gaussian and oscillating with twice the frequency of the motion along the limit cycle. Moreover the major and minor axes of the distribution oscillate as well as rotate during the motion.

This, as far as we know, represents the first quantisation of limit cycles in a dissipative system.

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Appendix 1

The Ito-Langevin equation

$$d\alpha^{(i)}(\tau) = f_i(\alpha^{(1)}(\tau), \alpha^{(2)}(\tau)) d\tau + e_i \alpha^{(i)}(\tau) dW_i(\tau)$$

formally has the solution

$$\alpha^{(i)}(\tau) = \alpha^{(i)}(0) + \int_0^\tau f_i(\alpha^{(1)}(s), \alpha^{(2)}(s)) ds + e_i \int_0^\tau \alpha^{(i)}(s) dW_i(s). \tag{A1.1}$$

The Ito integral $\int_0^\tau \alpha^{(i)}(s) dW_i(s)$ is defined as the limit of

$$\sum_{j=1}^n \alpha^{(i)}(S_{j-1})(W_i(s_j) - W_i(s_{j-1})) \tag{A1.2}$$

as $n \rightarrow \infty$ and

$$0 = s_0 \leq s_1 \leq s_2 \leq \dots \leq s_n = \tau. \tag{A1.3}$$

Moreover

$$\langle W_i(s_j) - W_i(s_{j-1}) \rangle = 0 \tag{A1.4}$$

$$\langle (W_i(s_j) - W_i(s_{j-1}))^2 \rangle = |s_j - s_{j-1}|. \tag{A1.5}$$

Of course $(W_i(s_j) - W_i(s_{j-1}))$ is a Gaussian variable. We shall obtain from (A1.1) a solution of the SDE whose mean and variance are correctly given up to $O(\tau^2)$. Hence correction terms will be $O(\tau^{5/2})$. The Euler method gives the mean and variance up to $O(\tau)$. The approximations are successively obtained by iterating (A1.1), i.e. the last

approximation found is substituted in the RHS of (A1.1) in order to generate the next approximation. We shall now give all the stages of approximation:

$$\alpha^{(i)}(\tau) = \alpha^{(i)}(0) + O(\tau^{1/2}). \tag{A1.6}$$

A single power of a Wiener increment, $(W(\tau) - W(0))$, gives a contribution of $O(\tau^{1/2})$ to $\alpha^{(i)}(\tau)$. By this we mean that the variance will have a direct contribution of $O(\tau)$ from such a term and a contribution proportional to $O(\tau^{1/2})$ from any non-trivial cross term which has it as a factor. There is, of course, no contribution to the mean. The trivial approximation gives the mean and variance accurate to $O(1)$.

The next approximation is

$$\alpha^{(i)}(\tau) = \alpha^{(i)}(0) + e_i \alpha^{(i)}(0) W_i(\tau) + O(\tau). \tag{A1.7}$$

The mean and variance are accurate to $O(\tau^{1/2})$. Since there is no contribution to the mean and variance to $O(\tau^{1/2})$ in (A1.7) it is no better than (A1.6).

Proceeding as before,

$$\begin{aligned} \alpha^{(i)}(\tau) = & \alpha^{(i)}(0) + f_i(\{\alpha^{(j)}(0)\})\tau + e_i \alpha^{(i)}(0) W_i(\tau) \\ & + e_i^2 \alpha^{(i)}(0) \int_0^\tau W_i(s) dW_i(s) + O(\tau^{3/2}). \end{aligned} \tag{A1.8}$$

In the mean square limit

$$\int_0^\tau W_i(s) dW_i(s) = \frac{1}{2}[(W_i(\tau))^2 - \tau]. \tag{A1.9}$$

Hence

$$\left\langle \int_0^\tau W_i(s) dW_i(s) \right\rangle = 0 \tag{A1.10}$$

$$\left\langle \left(\int_0^\tau W_i(s) dW_i(s) \right)^2 \right\rangle = \frac{1}{2}\tau^2. \tag{A1.11}$$

In this way we can show

$$\begin{aligned} \alpha^{(i)}(\tau) = & \alpha^{(i)}(0) + f_i(\underline{\alpha}(0))\tau + e_i \alpha^{(i)}(0) W_i(\tau) \\ & + \sum_{j=1}^2 f_{ij}(\underline{\alpha}(0)) e_j \alpha^{(j)}(0) \int_0^\tau W_j(s) ds \\ & + e_i f_i(\underline{\alpha}(0)) \int_0^\tau s dW_i(s) + e_i^2 \alpha^{(i)}(0) \int_0^\tau W_i(s) dW_i(s) \\ & + e_i^3 \alpha^{(i)}(0) \int_0^\tau dW_i(s) \int_0^s W_i(s') dW_i(s') \\ & + \frac{1}{2} \sum_{j=1}^2 f_{ij}(\underline{\alpha}(0)) f_j(\underline{\alpha}(0)) \tau^2 + O(\tau^{5/2}) \end{aligned} \tag{A1.12}$$

where

$$f_{ij} \equiv \partial f_i / \partial \alpha_j. \tag{A1.13}$$

This algorithm gives the mean and variance correct up to $O(\tau^2)$ to state the properties of the stochastic terms in (A1.12). On adopting the convention that $W_i(0) = 0$ the stochastic integrals in (A1.12) can be evaluated to give

$$\int_0^\tau W_i(s) dW_i(s) = \frac{1}{2}((W_i(\tau))^2 - \tau) \tag{A1.14}$$

$$\int_0^\tau dW_i(s) \int_0^s W_i(s') dW_i(s') = \frac{1}{6}(W_i(\tau))^3 - \frac{1}{2}\tau W_i(\tau) \tag{A1.15}$$

$$\int_0^\tau s dW_i(s) = \tau W_i(\tau) - \int_0^\tau W_i(s) ds. \tag{A1.16}$$

$W_i(\tau)$ and $\int_0^\tau W_i(s) ds$ are partly correlated Gaussian random variables. All other stochastic quantities in (A1.12) can be evaluated in terms of these two quantities.

$$\langle (W_i(\tau))^2 \rangle = \tau \tag{A1.17}$$

$$\left\langle W_i(\tau) \int_0^\tau W_i(s) ds \right\rangle = \frac{1}{2}\tau^2 \tag{A1.18}$$

$$\left\langle \left(\int_0^\tau W_i(s) ds \right)^2 \right\rangle = \frac{1}{3}\tau^3 \tag{A1.19}$$

so the covariance matrix is

$$M = \begin{pmatrix} \tau & \frac{1}{2}\tau^2 \\ \frac{1}{2}\tau^2 & \frac{1}{3}\tau^3 \end{pmatrix}. \tag{A1.20}$$

We can find a matrix C such that

$$CC^\tau = M. \tag{A1.21}$$

This is the Cholesky decomposition of M . A suitable C is

$$C = \begin{pmatrix} \tau^{1/2} & 0 \\ \frac{1}{2}\tau^{3/2} & \tau^{3/2}/2\sqrt{3} \end{pmatrix}. \tag{A1.22}$$

If ξ_1 and ξ_2 are a pair of independent Gaussian random variables of zero mean and unit variance

$$\begin{pmatrix} W_i(x) \\ \int_0^\tau W_i(s) ds \end{pmatrix} = C \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \tag{A1.23}$$

Hence, computationally, for each time step two calls are made to a random number generator for each variable.

Appendix 2

If

$$\alpha^{(j)} = \alpha_x^{(j)} + i\alpha_y^{(j)} \quad j = 1, 2 \tag{A2.1}$$

(where $\alpha_x^{(j)}$ and $\alpha_y^{(j)}$ are real) then it is convenient to use the scaled variables

$$\bar{\alpha}_x^{(j)} = (2\hbar/\omega)^{1/2} \alpha_x^{(j)} \quad j = 1, 2 \tag{A2.2}$$

$$\bar{\alpha}_y^{(j)} = (2\omega\hbar)^{1/2} \alpha_y^{(j)}. \tag{A2.3}$$

In terms of these quantities the diffusion matrix D has the following non-zero elements:

$$D_{11} = \varepsilon(\bar{\alpha}_x^{(1)} + \bar{\alpha}_y^{(1)}\omega^{-1})^2 \quad (\text{A2.4})$$

$$D_{22} = \varepsilon(\bar{\alpha}_x^{(2)} - \bar{\alpha}_y^{(2)}\omega^{-1})^2 \quad (\text{A2.5})$$

$$D_{33} = \varepsilon(\omega\bar{\alpha}_x^{(1)} - \bar{\alpha}_y^{(1)})^2 \quad (\text{A2.6})$$

$$D_{44} = \varepsilon(\omega\bar{\alpha}_x^{(2)} + \bar{\alpha}_y^{(2)})^2 \quad (\text{A2.7})$$

$$D_{31} = D_{13} = \varepsilon((\bar{\alpha}_y^{(1)})^2\omega^{-1} - \omega(\bar{\alpha}_x^{(1)})^2) \quad (\text{A2.8})$$

$$D_{42} = D_{24} = \varepsilon(\omega(\bar{\alpha}_x^{(2)})^2 - (\bar{\alpha}_y^{(2)})^2\omega^{-1}). \quad (\text{A2.9})$$

Moreover the drift A is given as follows:

$$A_1 = -(-\bar{\alpha}_y^{(1)}\omega^{-1} + \frac{1}{2}\gamma\omega^{-1}\bar{\alpha}_x^{(1)} - \lambda[\bar{\alpha}_y^{(2)}(\omega(\bar{\alpha}_x^{(1)})^2 - (\bar{\alpha}_y^{(1)})^2\omega^{-1}) + 2\omega\bar{\alpha}_x^{(1)}\bar{\alpha}_y^{(1)}\bar{\alpha}_x^{(2)}]) \quad (\text{A2.10})$$

$$A_2 = -(\bar{\alpha}_y^{(2)}\omega^{-1} + \frac{1}{2}\gamma\omega^{-1}\bar{\alpha}_x^{(2)} + \lambda[\bar{\alpha}_y^{(1)}(\omega(\bar{\alpha}_x^{(2)})^2 - (\bar{\alpha}_y^{(2)})^2\omega^{-1}) + 2\omega\bar{\alpha}_x^{(1)}\bar{\alpha}_x^{(2)}\bar{\alpha}_y^{(2)}]) \quad (\text{A2.11})$$

$$A_3 = -\{\omega\bar{\alpha}_x^{(1)} + \frac{1}{2}\gamma\omega^{-1}\bar{\alpha}_y^{(1)} + \lambda\omega[\bar{\alpha}_x^{(2)}(\omega^2(\bar{\alpha}_x^{(1)})^2 - (\bar{\alpha}_y^{(1)})^2) - 2\bar{\alpha}_x^{(1)}\bar{\alpha}_y^{(1)}\bar{\alpha}_y^{(2)}] \\ + \frac{1}{2}\mathcal{E}\omega^{-1}\cos(\omega_d/\omega)\tau\} \quad (\text{A2.12})$$

$$A_4 = -\{\frac{1}{2}\gamma\omega^{-1}\bar{\alpha}_y^{(2)} - \omega\bar{\alpha}_x^{(2)} - \lambda\omega[\bar{\alpha}_x^{(1)}(\omega^2(\bar{\alpha}_x^{(2)})^2 - (\bar{\alpha}_y^{(2)})^2) - 2\bar{\alpha}_y^{(1)}\bar{\alpha}_x^{(2)}\bar{\alpha}_y^{(2)}] \\ - \frac{1}{2}\mathcal{E}\omega^{-1}\cos(\omega_d/\omega)\tau\}. \quad (\text{A2.13})$$

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